Gyrokinetic limit of the 2D Hartree equation in a large magnetic field

Denis Périce
dperice@constructor.university

Joint work with

Nicolas Rougerie
nicolas.rougerie@ens-lyon.fr

CONSTRUCTOR UNIVERSITY

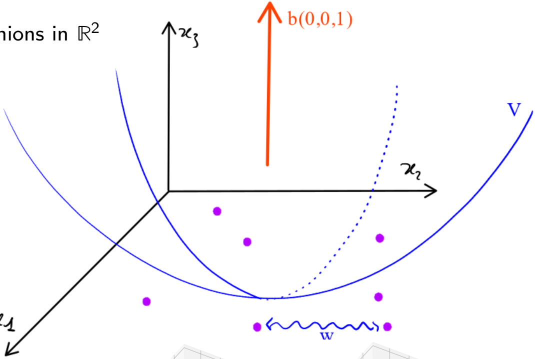
UMPA ENS DE LYON

Session on Mathematical and Theoretical Physics
02/08/2024

1 Context

Large system of spinless, non relativistic fermions in $\mathbb{R}^2$

- Homogeneous transverse magnetic field
- External potential $V : \mathbb{R}^2 \to \mathbb{R}$
- Radial interaction potential $w : \mathbb{R}^2 \to \mathbb{R}$



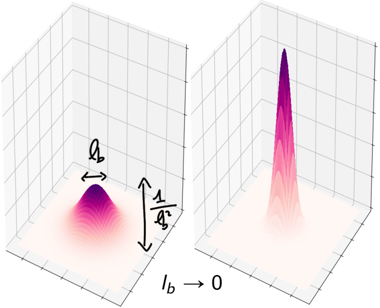
Motivation: Quantum Hall effect

Magnetic length: $l_b := \sqrt{\frac{\hbar}{b}}$

$\hbar$: reduced Planck's constant

Semi-classical/high magnetic field limit: $l_b \to 0$

Free ground state density on $\mathbb{R}^2$:



Goal: Effective dynamics

2 Model

Magnetic Laplacian

Unit system where $m = \frac{1}{2}, c = 1, q = 1,$

$$\mathscr{L}_b := (-i\hbar\nabla - bA)^2 = \sum_{n\in\mathbb{N}} 2\hbar b\left(n+\frac{1}{2}\right)\underbrace{\Pi_n}_{\text{projection on the } n^{th} \text{ Landau level}} \tag{1}$$

Vector potential in symmetric gauge:

$$A := \frac{X^\perp}{2} \implies \nabla \wedge A = (\partial_1, \partial_2, \partial_3) \wedge (A_1, A_2, 0) = (0, 0, 1) \tag{2}$$

where $X := (x_1, x_2)$ is the position operator.

Fermionic Density Matrix (FDM): $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ such that $\mathrm{Tr}\,(\gamma) = 1, 0 \leqslant \gamma \leqslant \dfrac{1}{N}$

Physical density: $\rho_\gamma : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}_+ \\ x & \mapsto & \gamma(x,x) \end{array}$

Hartree equation:

$$i l_b^2 \partial_t \gamma = [\mathscr{L}_b + V + w \star \rho_\gamma, \gamma] \tag{H}$$

Time scale: $l_b^{-2} = \dfrac{b}{\hbar}$

Scaling: $\quad l_b \to 0, \quad \hbar b = \mathcal{O}(1), \quad N = \mathcal{O}\left(l_b^{-2}\right)$

Drift equation: Given a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$,

$$\partial_t \rho(t,z) + \nabla^\perp(V + w \star \rho(t))(z) \cdot \nabla_z \rho(t,z) = 0 \tag{D}$$

3 Main result

$\Gamma(\mu,\nu)$: set of couplings between probabilities $\mu,\nu \in \mathcal{P}(\mathbb{R}^2)$, 1-Wasserstein metric:

$$W_1(\mu,\nu) := \inf_{\pi\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|\, d\pi(x,y) \tag{3}$$

Theorem: *Convergence of densities*

Let γ be the solution of (H) given $\gamma(0)$ a FDM such that for some $p > 7$,

$$\mathrm{Tr}\left(\gamma(0)\left(\mathscr{L}_b + V + \frac{1}{2} w * \rho_{\gamma(0)}\right)\right) \leqslant C, \quad \mathrm{Tr}\left(\gamma(0)\,|X|^p\right) \leqslant C \tag{4}$$

Let ρ solve (D). Assume $V, w \in W^{4,\infty}(\mathbb{R}^2)$ and $\nabla w \in L^1(\mathbb{R}^2)$, $w \in H^2(\mathbb{R}^2)$.
Then, $\forall t \in \mathbb{R}_+, \forall \varphi \in W^{1,\infty}(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma(t)} - \rho(t)\right)\right| \leqslant \widetilde{C}(t)\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(W_1\left(\rho_{\gamma(0)}, \rho(0)\right) + l_b^{\min\left(2\frac{p-7}{4p-7}, \frac{2}{7}\right)}\right) \tag{5}$$

Recap

$$i l_b^2 \partial_t \gamma = \left[\mathscr{L}_b + V + w \star \rho_\gamma, \gamma\right] \tag{H}$$

FDM: $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$, $\mathrm{Tr}(\gamma) = 1$, $0 \leqslant \gamma \leqslant \frac{1}{N}$, $\rho_\gamma(x) = \gamma(x,x)$

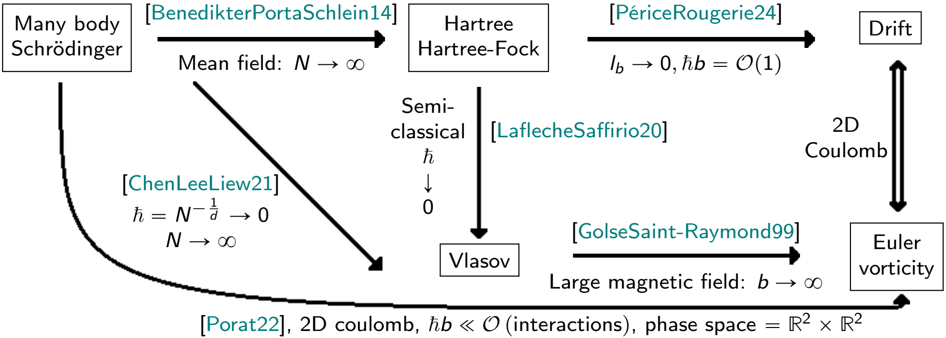
Scaling: $l_b \to 0$, $\quad \hbar b = \mathcal{O}(1)$, $\quad N = \mathcal{O}\left(l_b^{-2}\right)$

$$\partial_t \rho(t,z) + \nabla^\perp(V + w \star \rho(t))(z) \cdot \nabla_z \rho(t,z) = 0 \tag{D}$$

Challenges to overcome:

- Semi-classical phase space: $\mathbb{R}^2 \times \mathbb{N}$
- Controlling fast cyclotron motion
- Larger time scale

4 References



[PériceRougerie24] D.Périce N.Rougerie. "Gyrokinetic limit of the 2D Hartree equation in a large magnetic field". In: (2024). DOI: https://arxiv.org/abs/2403.19226.

[Porat22] I.B.Porat. "Derivation of Euler's equations of perfect fluids from von Neumann's equation with magnetic field". In: (2022). DOI: https://doi.org/10.48550/arXiv.2208.01158.

[ChenLeeLiew21] M.Liew L.Chen J.Lee. "Combined mean-field and semiclassical limits of large fermionic systems". In: *J Stat Phys* (2021). DOI: https://doi.org/10.1007/s10955-021-02700-w.

[LaflecheSaffirio20] L.Lafleche C.Saffirio. "Strong semiclassical limit from Hartree and Hartree-Fock to Vlasov-Poisson equation". In: *arXiv: Mathematical Physics* (2020). DOI: https://arxiv.org/abs/2003.02926.

[BenedikterPortaSchlein14] N.Benedikter M.Porta B.Schlein. "Mean–Field Evolution of Fermionic Systems". In: *Communications in Mathematical Physics* (2014). DOI: https://arxiv.org/abs/1305.2768.

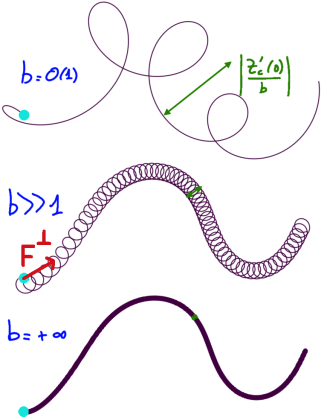
[GolseSaint-Raymond99] F.Golse L.Saint-Raymond. "The Vlasov–Poisson System with Strong Magnetic Field". In: *Journal de Mathématiques Pures et Appliquées* (1999). DOI: https://doi.org/10.1016/S0021-7824(99)00021-5.

5 Classical mechanics

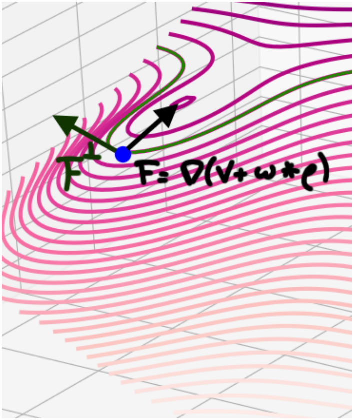
Newton's second law with constant homogeneous force field F:

$$Z'' = F + bZ'^{\perp} \implies Z(t) = \underbrace{\frac{|Z_c'(0)|}{b}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix}}_{\text{Cyclotron: } Z_c} + \underbrace{\frac{F^{\perp}}{b}t}_{\text{Drift: } Z_d \implies Z_d' = \frac{F^{\perp}}{b}} \quad \leftarrow \text{Drift time scale: } b \tag{6}$$

where we imposed $Z(0) = \frac{|Z_c'(0)|}{b}(1,0)$, $\quad Z'(0) = |Z_c'(0)|\,(0,1) + \frac{F^{\perp}}{b}$



Classical trajectories for different b



Level sets of $V + w \star \rho$

6 Semi-classical limit

Coherent state Let $\mathbf{z} \coloneqq z_1 + iz_2 \in \mathbb{C}$, and $z \coloneqq (z_1, z_2) \in \mathbb{R}^2$,

$$\varphi_{n,z}(x) \coloneqq \frac{i^n}{\sqrt{2\pi n!}\, l_b} \left(\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2} l_b} \right)^n e^{-\frac{|x-z|^2 - 2iz^\perp \cdot x}{4 l_b^2}} \tag{7}$$

Phase space projector:

$$\Pi_{n,z} \coloneqq |\varphi_{n,z}\rangle \langle \varphi_{n,z}| \,, \quad \Pi_z \coloneqq \sum_{n \in \mathbb{N}} |\varphi_{n,z}\rangle \langle \varphi_{n,z}| \tag{8}$$

satisfies

$$\frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \Pi_{n,z} dz = \Pi_n, \quad \Pi_z(x,y) = \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2 - 2i\left(x^\perp \cdot y + 2z^\perp \cdot (x-y)\right)}{4 l_b^2}}, \quad \nabla_z^\perp \Pi_z = \frac{1}{i l_b^2} [\Pi_z, X] \tag{*}$$

Let γ be a FDM

Phase space density

$$m_\gamma(n,z) \coloneqq \frac{1}{2\pi l_b^2} \mathrm{Tr}\left(\gamma \Pi_{n,z}\right)$$

Semi-classical density

$$\rho_\gamma^{sc}(z) \coloneqq \frac{1}{2\pi l_b^2} \mathrm{Tr}\left(\gamma \Pi_z\right) = \sum_{n \in \mathbb{N}} m_\gamma(n,z)$$

Truncated semi-classical density

$$\rho_\gamma^{sc, \leqslant M}(z) \coloneqq \sum_{n=0}^{M} m_\gamma(n,z)$$

Proposition: *Convergence of $\rho_\gamma^{sc,\leqslant M}$*

Let γ be a FDM, then $\forall \varphi \in L^\infty \cap H^1(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}^2} \varphi \left(\rho_\gamma - \rho_\gamma^{sc,\leqslant M} \right) \right| \leqslant C(\varphi)(M^{-\frac{1}{2}} + \underbrace{\sqrt{M} l_b}_{\text{Characteristic length inside MLL}}) \sqrt{\operatorname{Tr}(\gamma \mathscr{L}_b)} \tag{9}$$

We need $1 \ll M \ll l_b^{-2}$, higher Landau levels are controlled with the kinetic energy

$$\operatorname{Tr}(\gamma \mathscr{L}_b) = 2\hbar b \sum_{n \in \mathbb{N}} \left(n + \frac{1}{2} \right) \int_{\mathbb{R}^2} m_\gamma(n,z) dz \tag{10}$$

Proposition: *Gyrokinetic equation for the truncated semi-classical density*

Let $t \in \mathbb{R}_+, \gamma(t)$ be a FDM, $W \in W^{4,\infty}(\mathbb{R}^2)$ and assume

$$i l_b^2 \partial_t \gamma(t) = [\mathscr{L}_b + W, \gamma(t)], \quad \operatorname{Tr}(\gamma(t) \mathscr{L}_b) \leqslant C \tag{11}$$

then there exists a choice of $1 \ll M \ll l_b^{-2}$ such that $\forall \varphi \in L^1 \cap W^{1,\infty}(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \varphi \left(\partial_t \rho_{\gamma(t)}^{sc,\leqslant M} + \nabla^\perp W \cdot \nabla_z \rho_{\gamma(t)}^{sc,\leqslant M} \right) \underset{b \to \infty}{\longrightarrow} 0 \tag{12}$$

Convergence $\rho_\gamma^{sc,\leqslant M} \to \rho$:

- Dobrushin-type stability estimate for the limiting equation
- Use confinement for initial data

7 Central computation

We recall the dynamics and (*)

$$il_b^2\partial_t\gamma = \mathrm{Tr}\left(\mathscr{L}_b + W, \gamma\right), \quad \nabla_z^\perp \Pi_z = \frac{1}{il_b^2}\left[\Pi_z, X\right]$$

Evolution part

$$\begin{aligned}\partial_t\rho_\gamma^{sc}(z) &= \frac{1}{2\pi l_b^2}\mathrm{Tr}\left(\Pi_z\partial_t\gamma\right) = \frac{1}{2\pi l_b^2}\cdot\frac{1}{il_b^2}\mathrm{Tr}\left(\Pi_z\left[\mathscr{L}_b + W, \gamma\right]\right) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma\left[\Pi_z, \mathscr{L}_b + W\right]\right) \\ &= \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma\left[\Pi_z, W\right]\right)\end{aligned} \tag{13}$$

Spacial part

$$\begin{aligned}\nabla^\perp W(z)\cdot\nabla\rho_\gamma^{sc}(z) &= -\nabla W(z)\cdot\frac{1}{2\pi l_b^2}\mathrm{Tr}\left(\gamma\nabla_z^\perp\Pi_z\right) = -\frac{1}{2i\pi l_b^4}\nabla W(z)\cdot\mathrm{Tr}\left(\gamma\left[\Pi_z, X\right]\right) \\ &= -\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma\left[\Pi_z, \nabla W(z)\cdot X\right]\right)\end{aligned} \tag{14}$$

so

$$\partial_t\rho_\gamma^{sc}(z) + \nabla^\perp W(z)\cdot\nabla\rho_\gamma^{sc}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left(\gamma\left[\Pi_z, W - \nabla W(z)\cdot X\right]\right) \tag{15}$$

where

$$\left[\Pi_z, W - \nabla W(z)\cdot X\right](x,y) = \Pi_z(x,y)\left(W(y) - W(x) - \nabla W(z)\cdot(y-x)\right) \tag{16}$$

Thanks for your attention